Utility maximisation and indifference curves - an explanation

Part 1: Utility functions

Suppose a consumer wants to maximise their utility. Their utility depends on the quantities of two goods consumed.

This can be expressed with a utility function:

$$U = f(x, y)$$

where U is utility, x is the amount of good x consumed, and y is the amount of good y consumed.

A typical example of a utility function is the Cobb-Douglas function:

$$U = x^{\alpha}y^{\beta}$$

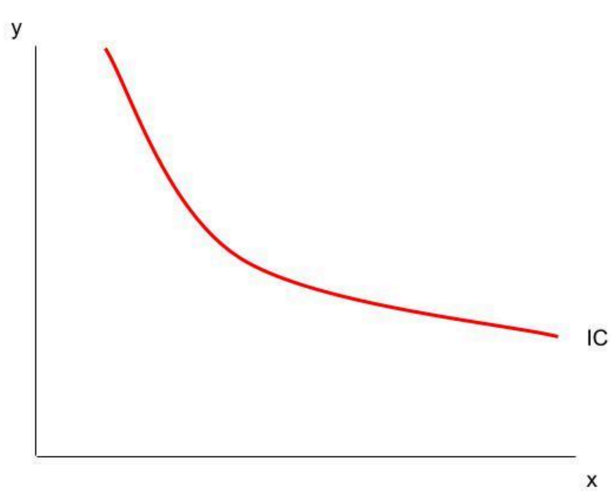
where typically α and β lie between 0 and 1.

Part 2: Indifference curves

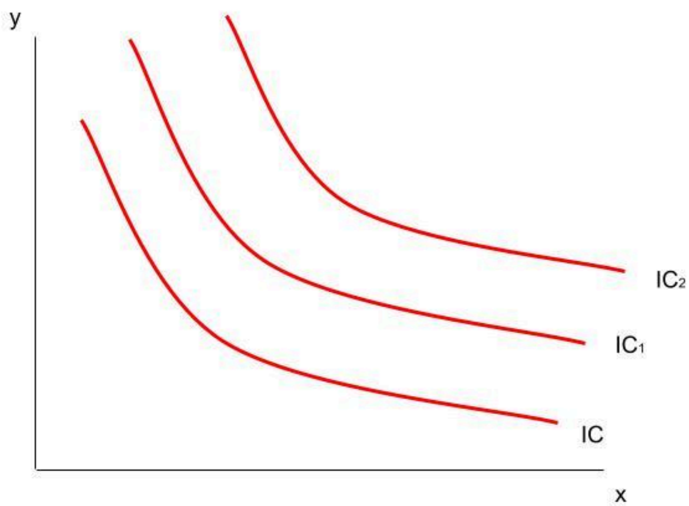
An indifference curve shows all combinations of amounts of two goods x and y that deliver the same utility.

For example, suppose $(x, y) = (4, 1), (2, 2), (1, 4)$ could all be a set of bundles that deliver the same utility.

An indifference curve may look like this:



We can draw different indifference curves to denote different levels of utility. With Cobb-Douglas preferences, a higher indifference curve denotes a higher utility level.



The slope of the indifference curve is called the marginal rate of substitution (MRS). This is:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

which can also be written as

$$-\frac{MU_x}{MU_y}$$

Deriving the MRS:
Start with the utility function $U = f(x, y)$ and differentiate with respect to x.

$$\frac{dU}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

Written by Tom Furber
Economics educator
tfurber.com

Note $\frac{dU}{dx} = 0$ as utility is constant along an indifference curve.

Rearrange:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \text{ which can also be written as } -\frac{MU_x}{MU_y}.$$

where MU_x refers to the marginal utility of good x

Part 3: Budget constraints

A consumer has a budget of say m. The total expenditure on goods x and y must fit within the consumer's budget, which creates a budget constraint:

$$p_x x + p_y y \leq m$$

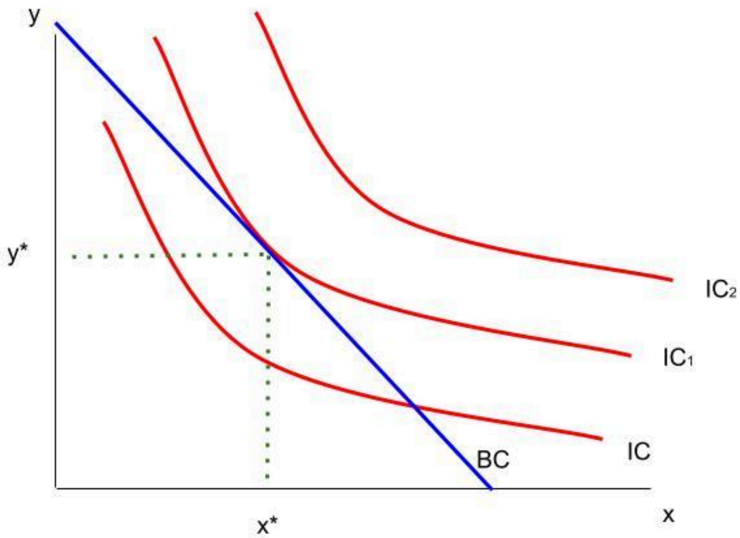
where p_x is the price of one unit of good x and p_y is the price of one unit of good y.

The budget constraint in (x,y) space is simply a downward-sloping line with a constant gradient.

For most utility maximisation problems, the budget constraint will hold with equality. Provided utility is strictly increasing in x and y, then if consumers are not spending their full budget, an increase in the amount of x or y will increase utility.

Part 4: Solving utility maximisation problems - graphical approach of comparing gradients

By layering the budget constraint and the indifference curves on top of each other, we can graphically show the utility maximising outcome.



The utility maximising outcome occurs where the indifference curve is tangent to the budget line. This occurs at $(x\ *, y\ *)$ on the diagram.

- Higher levels of utility, represented by indifference curves above IC1, would not be within the consumer's budget.
- Lower levels of utility, represented by indifference curves below IC1, do not maximise utility. The consumer can increase utility by moving onto a higher indifference curve, wile remaining within the budget.

This occurs where the MRS (in absolute value) equals the price ratio:

$$\frac{MU_x}{MU_y} = \frac{p_x}{p_y}$$

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Economics educator
tfurber.com

Tip

One way to remember this condition is that to maximise utility within the budget, the consumer chooses x and y such that the marginal utility per pound (or dollar) is equal across all goods. In other words:

$$\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$$

Example question

Maximise $U = xy$ subject to the budget constraint $x + 2y = 4$.

$$MRS = -\frac{MU_x}{MU_y} = -\frac{y}{x}$$

$$Price\ ratio = \frac{p_x}{p_y} = \frac{1}{2}$$

From the graph we know that the magnitude of the slope of the indifference curve equals the price ratio.

In other words the absolute value of the MRS equals the price ratio:

$$\frac{y}{x} = \frac{1}{2}$$

$$2y = x$$

Substitute into the budget constraint:

$$2x = 4;\ x = 2$$

Substitute $x = 2$ into the equation $2y = x$ gives $y = 1$.

Answer: $x = 2,\ y = 1.$

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Economics educator
tfurber.com

Part 5: Solving utility maximisation problems - mathematical approach via substitution

Another approach, which gets more involved if the powers on x and y are not integers, is a substitution process.

- Step 1: substitute the budget constraint into the utility function. This converts a multivariate constrained optimisation problem into a one-variable unconstrained optimisation problem.
- Step 2: differentiate with respect to the remaining variable and find the maximum.

Maximise $U = xy$ subject to the budget constraint $x + 2y = 4$.

Step 1: turn a two variable maximisation problem into a one variable problem. We can do this by substituting the budget constraint into the utility function.

The budget constraint can be rearranged $x = 4 - 2y$. Plug this into the utility function:

$$U = (4 - 2y)y = 4y - 2y^2$$

Differentiate and find the stationary point:

$$4 - 4y = 0$$
$$y = 1$$

Plug this into the budget constraint to find x:

$$x = 4 - 2y = 4 - 2(1) = 2$$

[Note we could prove this is a maximum by differentiating again and finding the second derivative is negative].

Answer: $x = 2,\ y = 1.$

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tfurber.com

Extension questions to consider:

- When does the rule that the MRS (in absolute value) equals the price ratio fail to generate the utility maximising outcome?
- What happens when the utility maximisation problem is solved for a more general Cobb-Douglas production function?

During an undergraduate degree, you may also learn the Lagrangian method for solving constrained optimisation problems. This method shows mathematically where the condition that "the (absolute value of the) MRS equals the price ratio" comes from.

Written by Tom Furber
Economics educator
tfurber.com